SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 1

SOLUTIONS

In the following problems, a "fake n-manifold" is a topological space M which is locally Euclidean. That is, for every point $p \in M$, there exists a neighborhood $U \subset M$ of p, an open set $V \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \to V$.

Problem 1. Let M be a connected topological n-manifold, and $C \subset M$ be a closed proper subset. Let M^f be the set

$$M^f = \left\{ (x, a) : x \in M, \text{ and } \begin{array}{ll} a = 0, & x \notin C \\ a = \pm 1, & x \in C \end{array} \right\}$$

Define a topology on M^f as being generated by two types of open sets from derived from the topology on M: If $U \subset M$ is open, let

- $U^{\#} = \{(x, 0) : x \in U \setminus C\} \cup \{(x, 1) : x \in C \cap U\}$, and
- $U^{\flat} = \{(x, 0) : x \in U \setminus C\} \cup \{(x, -1) : x \in C \cap U\}$.

Show that, with the topology generated by sets of the form $U^{\sharp}$ and $U^{\flat}$, M^f is a fake n-manifold, but not a manifold. What happens if $C = M$?

Solution. We first claim that M^f is locally Euclidean. Fix $p^f = (p, \sigma) \in M^f$. Since M is an n-manifold, there exists some $U \subset M$ and some map $\varphi : U \to \mathbb{R}^n$ which is a homeomorphism onto its image.

Case: $p \notin C$. In this case $U_0 = U \setminus C$ still contains p, and $\varphi|_{U_0}$ is a homeomorphism onto its image, since it is the restriction of a homeomorphism. Therefore, $U_0 \times \{0\} \subset M^f$ is open and $\varphi^f(x, \sigma) = \varphi(x)$ is a homeomorphism onto its image, since $U_0^{\sharp} = U_0^{\flat}$ projects to U_0.

Case: $p \in C$. In this case, $\sigma = \pm 1$. Without loss of generality consider $\sigma = 1$. With U as above, we claim that the set $U^{\sharp}$ is naturally homeomorphic to U (which is itself homeomorphic to an open subset of $\mathbb{R}^n$). Indeed, define $g_{\sharp} : U \to U^{\sharp}$ by $g(p) = (p, 1)$. Since the topology on $U^{\sharp}$ is generated by sets of the form $V^{\sharp} \cap U^{\sharp}$ and $V^{\flat} \cap U^{\sharp}$, where $V \subset M$ is open, we must show that their projection to M is open. Indeed, the M-projection of $V^{\sharp} \cap U^{\sharp}$ is $V \cap U$, and the M-projection of $V^{\flat} \cap U^{\sharp}+$ is $(V \setminus C) \cap U$. Since C is closed, this set is open. Conversely, if $V \subset U$ is open, then $V^{\sharp} = g_{\sharp}(V) \subset U^{\sharp}$ is open. Thus, we conclude that $g_{\sharp}$ is a homeomorphism, and M^f is locally Euclidean.

Finally, we claim that M^f is not Hausdorff. Indeed, pick any $p \in C$ such that every neigbhorhood U of p is not contained in C. Such a point must exist, otherwise every point of C is an interior point, and C is both open and closed. Since M is connected, C would be trivial or empty, which violates that it is proper. Now, with such a p in hand, notice that $(p, 1), (p, -1) \in M^f$ are points, and any neighborhood of $(p, 1)$ must be of the form $U_1^{\sharp}$, where U_1 is a neighborhood of p, while any neighborhood of $(p, -1)$ must be form $U_2^{\flat}$, where U_2 is a neighborhood of p. Then $U_1 \cap U_2$ is a neighborhood of p, and by choice of p, $U_1 \cap U_2 \cap (M \setminus C) \neq \emptyset$. Therefore, $U_1^{\sharp} \cap U_2^{\flat}$ contains $(U_1 \cap U_2 \cap (M \setminus C)) \times \{0\}$, which is nonempty. Therefore, M^f is not Hausdorff. Note that if $C = M$, then $M^{\flat}$ is exactly $M \times \{-1, 1\}$, which is a manifold. ∎

Problem 2. Give an example of a Hausdorff fake n-manifold which is not a manifold (and justify why it is not a manfiold).

Solution. Let S be any set of uncountable cardinality, and give it the discrete topology. Then $M^{\#} = \mathbb{R} \times S$ is locally Euclidean (it is locally homeomorphic to $\mathbb{R}$. However, $\{0\} \times S$ is exactly S with the discrete topology, which does not have a countable basis. Therefore, $M^{\#}$ is not a manifold, since it is not second countable. □

Problem 3. Show that $S^n = \left\{x \in \mathbb{R}^{n+1} : ||x|| = 1\right\}$ has a canonical smooth n-manifold structure by explicitly finding a smooth atlas and showing the atlas is smooth.

Solution. We use the *stereographic projections* to construct our charts. Let $\varphi_1(x_1, \ldots, x_{n+1}) = \frac{1}{1-x_{n+1}}(2x_1, \ldots, 2x_n)$ be defined from $S^n \setminus \{(0, \ldots, 0, 1)\}$ to $\mathbb{R}^n$. We claim that φ_1 is a homeomorphism onto its image. Indeed, its inverse can be computed explicitly, and is the rational function

$$\varphi_1^{-1}(y_1, \ldots, y_n) = \frac{1}{4 + \sum {y_i}^2}\left(4y_1, \ldots, 4y_n, -4 + \sum y_i^2\right).$$

Similarly, we let $\varphi_2(x_1, \ldots, x_{n+1}) = \frac{1}{1+x_{n+1}}(2x_1, \ldots, 2x_n)$ be defined on $S^n \setminus \{(0, \ldots, 0, -1)\}$. Then

$$\varphi_2^{-1}(y_1, \ldots, y_n) = \frac{1}{4 + \sum {y_i}^2}\left(4y_1, \ldots, 4y_n, 4 - \sum y_i^2\right).$$

To show that φ_1 and φ_2 form a smooth atlas, we must confirm that $\varphi_1 \circ \varphi_2^{-1}$ is a diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ to itself. Indeed,

$$\varphi_1 \circ \varphi_2^{-1}(y_1, \ldots, y_n) = \frac{4 + \sum y_i^2}{4 \sum y_i^2}(y_1, \ldots, y_n),$$

which is indeed a well-defined diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ to itself. □

Problem 4. Show that if M is a smooth m-manifold and N is a smooth n-manifold, then $M \times N$ has a canonical smooth $(m + n)$-manifold structure.

Solution. First, notice that if M and N are both Hausdorff and second countable, then so is $M \times N$. We now construct a smooth atlas on $M \times N$. Indeed, fix atlases $\mathcal{A}_M$ for M and $\mathcal{A}_N$ for N, respectively. Then if $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$, let $\varphi \times \psi : U \times V \to \mathbb{R}^{n+m}$ be defined by $\varphi \times \psi(x, y) = (\varphi(x), \psi(y))$.

We claim that the collection of pairs $(\varphi \times \psi, U \times V) \in \mathcal{A}_M \times \mathcal{A}_N$ is a smooth atlas. Indeed, notice that $U_1 \times V_1$ and $U_2 \times V_2$ intersect if and only if $U_1 \cap U_2 \neq \emptyset$ and $V_1 \times V_2 \neq \emptyset$. Furthermore, the corresponding maps satisfy:

$$(\varphi_2 \times \psi_2) \circ (\varphi_1 \times \psi_1)^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1}),$$

which is verified directly by computation. Since the maps repsect the decomposition into direct products, it is C^∞ since each component is C^∞. Therefore, we have constructed a smooth atlas. □

Problem 5. Show that $\mathbb{R}^2 \setminus \{0\}$, $A = \left\{x \in \mathbb{R}^2 : 1 < ||x|| < 2\right\}$ and $S^1 \times (0, 1)$ are all diffeomorphic with their standard smooth structures.

Solution. We construct a diffeomorphism between $S^1 \times (0, 1)$ and the first two manifolds. First, define $F : S^1 \times (0, 1) \to \mathbb{R}^2 \setminus \{0\}$ as

$$F(x,t) = \frac{tx}{1-t}.$$

Then F is invertible and $F^{-1}(y) = (y/||y||, ||y||/(1+||y||))$. Since $||\cdot||$ is differentiable except at $y=0$, F is a diffeomorphism between $S^1 \times (0,1)$ and $\mathbb{R}^2 \setminus \{0\}$.

Now we construct a diffeomorphism between $S^1 \times (0,1)$ and A. Let $G(x,t) = (t+1)x$. It is clear that G is smooth and that $G^{-1}(y) = (y/||y||, ||y|| - 1)$. Therefore, G is a diffeomorphism ☐

Problem 6. Let X denote the boundary of the unit square in $\mathbb{R}^2$. Prove or find a counterexample:

(1) X is a topological 1-manifold.
(2) There exists a smooth structure on X.
(3) There exists a smooth structure on X such that the inclusion of X into $\mathbb{R}^2$ is C^∞.
(4) There exists a smooth structure on X such that the inclusion of X into $\mathbb{R}^2$ is an immersion.

Solution.

(1) and (2) **These exist.** Choose any homeomorphism between ∂I^2 and S^1. For instance, define $F : \partial I^2 \to S^1$ by $\varphi(x) = x/||x||$ (here, we take $I = [-1/2, 1/2]$ rather than $[0,1]$). If $U \subset S^1$ and $\varphi : U \to \mathbb{R}$ is a smooth chart for S^1, let $V = F^{-1}(U)$ and $\tilde{\varphi}(x) = \varphi \circ F$. We claim that the collection of charts $\tilde{\varphi}$, where φ is a smooth chart for S^1 is a smooth structure. Indeed, the transition functions are exactly

$$\tilde{\varphi_1} \circ \tilde{\varphi_2}^{-1} = (\varphi_1 \circ F) \circ (\varphi_2 \circ F)^{-1} = \varphi_1 \circ \varphi_2^{-1}.$$

These are clearly C^∞, so we have constructed a smooth structure. Furthermore, the map F is a diffeomorphism when using these charts, since when using the charts φ and $\tilde{\varphi}$ on S^1 and ∂I^2, respectively, the map F is represented by the identity.

(3) **This exists.** We refine the choice of F from the previous construction. Indeed, we define F piecewise on each segment of the boundary. We define it on $\{1/2\} \times [-1/2, 1/2]$, the definition on the remaining components is clear. Let $\psi : (-1,1) \to [-1/2, 1/2]$ be any C^∞ function such that
(a) $\psi'(t) > 0$ when $t \in (-1/2, 1/2)$,
(b) $\psi^{(k)}(t) = 0$ for all $k \geq 1$ when $|t| \geq 1/2$,
(c) $\psi(-1/2) = -1/2$, and
(d) $\psi(1/2) = 1/2$.

Notice that ψ must be invertible on $[-1/2, 1/2]$ (since it is increasing), which we use in the following (we denote $(\psi|_{[-1/2,1/2]})^{-1}$ by ψ^{-1} for simplicity of notation). Give ∂I^2 the smooth structure, where the charts are given locally by maps $\varphi(1/2, t) = \psi^{-1}(t)$ (and similarly for other edges of the square). At the corner point $(-1/2, -1/2)$, we define

$$\varphi(s,t) = \begin{cases} \psi^{-1}(s) & t = -1/2 \\ -1 - \psi^{-1}(t) & s = -1/2 \end{cases}$$

on the set $(\{-1/2\} \times [-1/2, 1/2)) \cup ([-1/2, 1/2) \times \{-1/2\})$. Notice that the image of $[-1/2, 1/2)$ under ψ^{-1} is $[-1/2, 1/2)$, and the image under $-1 - \psi^{-1}$ is $(-3/2, -1/2]$. Furthermore, the map is well defined at $(-1/2, -1/2)$, since $-1/2 = -1 - (-1/2)$. Hence ψ^{-1} is a homeomorphism from the union of the edges meeting at $(-1/2, -1/2)$ and $(-3/2, 1/2)$. Such a family clearly forms a smooth atlas when using similar definitions at other corner points, since the intersections consist only of open intervals, and transition maps are given by compositions of ψ, ψ^{-1} and the inversion of the interval, $I(x) = -x$, which are C^∞ on the interiors.

Then since ψ is C^∞, on the interior of the edges, the inclusion is clearly C^∞. At the corner point, observe that the inclusion is determined by ψ, and since all derivatives vanish, the inclusion is C^∞.

(4) **This does not exist.** Assume that the inclusion of ∂I^2 is an immersion. Choose a local chart centered at a corner point. Without loss of generality, assume that U is a neighborhood of $(1/2, 1/2)$ in ∂I^2, and $\varphi : U \to \mathbb{R}$ is a chart such that $\varphi(1/2, 1/2) = 0$. Then φ is invertible, and since the inclusion is an immersion, $\varphi^{-1} : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ is C^∞. Without loss of generality, assume that negative values of t are taken to the horizontal edge near $(1/2, 1/2)$ and positive values are taken to the vertical edge. Then for each $\varepsilon > \delta > 0$, by the Mean Value Theorem, there exists some $t \in (0, \delta)$ such that $(\varphi^{-1})'(t)$ is a vertical vector (ie, has 0 as its first component). Similarly, for every $-\varepsilon < \delta < 0$, there exists some $t \in (0, \delta)$ such that $(\varphi^{-1})'(t)$ is a horizontal vector (ie, has 0 as its second component). Therefore, $(\varphi^{-1})'(0)$ must have zero in both components, since it is a limit of such vectors. Therefore, $(\varphi^{-1})'(0) = 0$, which contradicts that the inclusion is an immersion.

□

Linear algebra and vector calculus review

Problem 7. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism, and assume that there exists a $v \in \mathbb{R}^n$ such that v is an eigenvector of $DF(x)$ with real eigenvalue for every $x \in \mathbb{R}^n$. Show that the lines $L(x) = \{x + tv : t \in \mathbb{R}\}$ are *equivariant*: $F(L(x)) = L(F(x))$.

Solution. Since F is a diffeomorphism and $DF(p)^{-1} = DF^{-1}(F(p))$, it follows that if v is an eigenvector with eigenvalue $\lambda(p)$ for every matrix $DF(p)$, the it is also an eigenvector of $DF^{-1}(q)$ of eigenvalue λ^{-1} for every $q \in \mathbb{R}^k$. Notice that 0 cannot be an eigenvalue since DF must be invertible.

Hence it suffices to show that $F(L(x)) \subset L(F(x))$, as the opposite inclusion will follow from the analysis on F^{-1}. Let $\gamma_x : \mathbb{R} \to \mathbb{R}^n$ be the map $\gamma_x(t) = x + tv$, so that $L(x)$ is the image of γ_x. Then $F(L(x))$ is the image of $F \circ \gamma$. But the derivative of $F \circ \gamma$ at t is

$$DF(\gamma(t))\gamma'(t) = DF(\gamma(t))v = \lambda(\gamma(t))v,$$

which is a multiple of v by assumption. Therefore,

$$F(\gamma(t)) = F(x) + \int_0^t DF(\gamma(s))\gamma'(s)\, ds = F(x) + \int_0^t \lambda(\gamma(s))\, ds \cdot v.$$

Hence, $F(L(x)) \subset L(F(x))$, as claimed. □

Problem 8. Let V and W be (real) finite-dimensional vector spaces and $\mathrm{End}(V, W)$ be the set of linear transformations from V to W.

(1) Show that $\mathrm{End}(V, W)$ is a real vector space.
(2) With fixed bases for V and W, find an isomorphism between $\mathrm{End}(V, W)$ and $M(m, n)$, the set of $m \times n$ matrices, where $m = \dim(V)$ and $n = \dim(W)$.
(3) If $V_0 \subset V$ is a subspace of V, let $\mathrm{Ann}(V_0) \subset \mathrm{End}(V, W)$ be the *annihilator* of V_0. That is, the set of $\varphi \in \mathrm{End}(V, W)$ such that $\varphi(v) = 0$ for all $v \in V_0$. Show that $\mathrm{Ann}(V_0)$ is a vector subspace of $\mathrm{End}(V, W)$, then find and prove a formula for $\dim(\mathrm{Ann}(V_0))$ in terms of $\dim(V)$, $\dim(W)$ and $\dim(V_0)$. [*Hint*: It might be useful to think about it as matrices using the previous part]
(4) * Find a canonical isomorphism between $V^* \otimes W$ and $\mathrm{End}(V, W)$, and prove it is an isomorphism. Construct a projection $\pi : V^* \otimes W \to V_0^* \otimes W$ such that $\mathrm{Ann}(V_0) = \ker \pi$, and prove that it is a projection, and that the kernel is as described. Deduce the formula for $\dim(\mathrm{Ann}(V_0))$ using π, as well.

Solution.

(1) Let $\varphi, \psi \in \mathrm{End}(V, W)$. Then $\varphi + \psi$ is also a linear transformation, and hence in $\mathrm{End}(V, W)$. Similarly, if $c \in \mathbb{R}$, $(c\varphi)(v) = c\varphi(v)$ is a linear transformation. Hence $\mathrm{End}(V, W)$ is a real vector space.
(2) Let $v_1, \ldots, v_m$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. Then if $\varphi \in \mathrm{End}(V, W)$, let a_{ij} be the w_j-component of $\varphi(v_i)$. Then $A = (a_{ij})$ is an $m \times n$-matrix, and the map $\varphi \mapsto A$ is a homomorphism. Furthermore, it is an isomorphism, since if A is the zero matrix, $\varphi \equiv 0$, as it vanishes on a basis. Furthermore, if A is any matrix, then one may define a linear transformation via:

$$\varphi(t_1 v_1 + \cdots + t_m v_m) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} t_i w_j$$

(3) Observe that if $\varphi, \psi \in \text{Ann}(V_0)$, then if $v \in V_0$, $(\varphi + \psi)(v) = \varphi(v) + \psi(v) = 0$. Similarly, $c\varphi(v) = 0$. Therefore, $\text{Ann}(V_0)$ is a vector subspace. Assume that $\dim(V_0) = k$, and that $v_1, \ldots, v_k$ form a basis of V_0. Then a matrix A corresponds to an element of $\text{Ann}(V_0)$ if it takes the following form:

$$\begin{pmatrix} 0 & \ldots & 0 & * & \ldots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & * & \ldots & * \end{pmatrix}$$

where the block of 0's is an $k \times n$ block. Therefore, the dimension of $\text{Ann}(V_0)$ is $(m-k) \cdot n$.

(4) Given an element $\lambda \otimes w \in V^* \otimes W$, let $\varphi_{\lambda,w}(v) = \lambda(v)w$. This gives a homomorphism from $V^* \otimes W$ to $\text{End}(V, W)$, since the map $(\lambda, w) \mapsto \varphi_{\lambda,w}$ is bilinear. The map is an isomorphism. Let $\pi : V^* \otimes W \to V_0^* \otimes W$ be the restriction map, $\pi(\varphi \otimes w) = (\varphi|_{V_0} \otimes w)$. Then π is a surjective homomorphism, and it is clear that $\ker \pi = \text{Ann}(V_0)$, since $\text{Ann}(V_0)$ are exactly the transformations which restrict to the trivial transformation on V_0. Hence $\dim(\text{Ann}(V_0)) = \dim(V^* \otimes W) - \dim(V_0^* \otimes W) = m \cdot n - k \cdot n = n(m-k)$, as clained.

□